The Finite Difference Method

10.1 Hyperbolic Equations

Wave Equation

As an example of a hyperbolic partial differential equation, we consider the wave equation

(1) $$u_{tt}(x, t) = c^2 u_{xx}(x, t) \quad \text{for } 0 < x < a \text{ and } 0 < t < b,$$

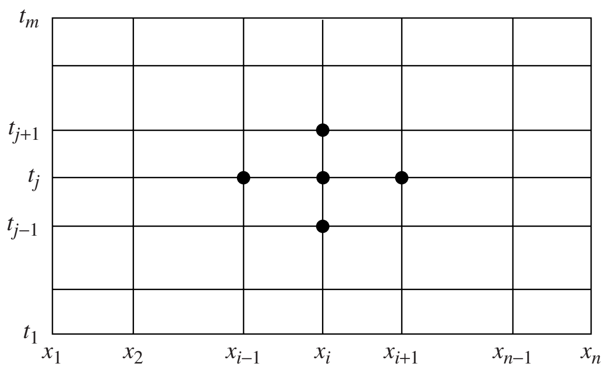

Figure 10.4 The grid for solving $u_{tt}(x,t) = c^2 u_{xx}(x,t)$ over R.

with the boundary conditions

$$
\begin{aligned}
&u(0,t) = 0 \quad \text{and} \quad u(a,t) = 0 && \text{for } 0 \le t \le b, \\
&u(x,0) = f(x) && \text{for } 0 \le x \le a, \\
&u_t(x,0) = g(x) && \text{for } 0 < x < a.
\end{aligned} \tag{2}
$$

The wave equation models the displacement u of a vibrating elastic string with fixed ends at $x = 0$ and $x = a$. Although analytic solutions to the wave equation can be obtained with Fourier series, we use the problem as a prototype of a hyperbolic equation.

Derivation of the Difference Equation

Partition the rectangle $R = \{(x,t) : 0 \le x \le a, 0 \le t \le b\}$ into a grid consisting of $n-1$ by $m-1$ rectangles with sides $\Delta x = h$ and $\Delta t = k$, as shown in Figure 10.4. Start at the bottom row, where $t = t_1 = 0$ and the solution is known to be $u(x_i, t_1) = f(x_i)$. We shall use a difference-equation method to compute approximations

$$\{u_{i,j} : i = 1, 2, \ldots, n\} \text{ in successive rows} \quad \text{for } j = 2, 3, \ldots, m.$$

The true solution value at the grid points is $u(x_i, t_j)$.

The central-difference formulas for approximating $u_{tt}(x,t)$ and $u_{xx}(x,t)$ are

$$u_{tt}(x,t) = \frac{u(x, t+k) - 2u(x,t) + u(x, t-k)}{k^2} + \boldsymbol{O}(k^2) \tag{3}$$

and

$$u_{xx}(x,t) = \frac{u(x+h, t) - 2u(x,t) + u(x-h, t)}{h^2} + \boldsymbol{O}(h^2). \tag{4}$$

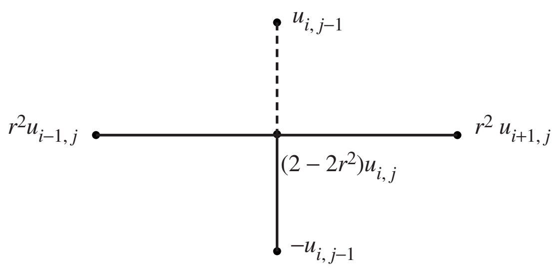

Figure 10.5 The wave equation stencil.

The grid spacing is uniform in every row: $x_{i+1} = x_i + h$ (and $x_{i-1} = x_i - h$); and it is uniform in every column: $t_{j+1} = t_j + k$ (and $t_{j-1} = t_j - k$). Next, we drop the terms $\boldsymbol{O}(k^2)$ and $\boldsymbol{O}(h^2)$ and use the approximation $u_{i,j}$ for $u(x_i, t_j)$ in equations (3) and (4), which in turn are substituted into (1); this produces the difference equation

(5) $$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = c^2 \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2},$$

which approximates equation (1). For convenience, the substitution $r = ck/h$ is introduced in (5), and we obtain the relation

(6) $$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = r^2(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}).$$

Equation (6) is employed to find row $j+1$ across the grid, assuming that approximations in both rows j and $j-1$ are known:

(7) $$u_{i,j+1} = (2 - 2r^2)u_{i,j} + r^2(u_{i+1,j} + u_{i-1,j}) - u_{i,j-1},$$

for $i = 2, 3, \ldots, n-1$. The four known values on the right side of equation (7), which are used to create the approximation $u_{i,j+1}$, are shown in Figure 10.5.

Caution must be taken when using formula (7). If the error made at one stage of the calculations is eventually dampened out, the method is called *stable*. To guarantee stability in formula (7), it is necessary that $r = ck/h \le 1$. There are other schemes, called *implicit* methods, that are more complicated to implement, but do not have stability restrictions for r.

Starting Values

Two starting rows of values corresponding to $j = 1$ and $j = 2$ must be supplied in order to use formula (7) to compute the third row. Since the second row is not usually given, the boundary function $g(x)$ is used to help produce starting approximations in the second row. Fix $x = x_i$ at the boundary and apply Taylor's formula of order 1 for expanding $u(x, t)$ about $(x_i, 0)$. The value $u(x_i, k)$ satisfies

(8) $$u(x_i, k) = u(x_i, 0) + u_t(x_i, 0)k + \boldsymbol{O}(k^2).$$

Then use $u(x_i, 0) = f(x_i) = f_i$ and $u_t(x_i, 0) = g(x_i) = g_i$ in (8) to produce the formula for computing the numerical approximations in the second row:

$$u_{i,2} = f_i + kg_i \quad \text{for } i = 2,\ 3,\ \ldots,\ n-1. \tag{9}$$

Usually, $u(x_i, t_2) \neq u_{i,2}$, and such errors introduced by formula (9) will propagate throughout the grid and will not be dampened out when the scheme in (7) is implemented. Hence it is prudent to use a very small step size for k so that the values for $u_{i,2}$ given in (9) do not contain a large amount of truncation error.

Often, the boundary function $f(x)$ has a second derivative $f''(x)$ over the interval. In this case we have $u_{xx}(x, 0) = f''(x)$, and it is beneficial to use the Taylor formula of order $n = 2$ to help construct the second row. To do this, we go back to the wave equation and use the relationship between the second-order partial derivatives to obtain

$$u_{tt}(x_i, 0) = c^2 u_{xx}(x_i, 0) = c^2 f''(x_i) = c^2 \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} + \boldsymbol{O}(h^2). \tag{10}$$

Recall that Taylor's formula of order 2 is

$$u(x, k) = u(x, 0) + u_t(x, 0)k + \frac{u_{tt}(x, 0)k^2}{2} + \boldsymbol{O}(k^3). \tag{11}$$

Applying formula (11) at $x = x_i$, together with (9) and (10), we get

$$u(x_i, k) = f_i + kg_i + \frac{c^2k^2}{2h^2}(f_{i+1} - 2f_i + f_{i-1}) + \boldsymbol{O}(h^2)\boldsymbol{O}(k^2) + \boldsymbol{O}(k^3). \tag{12}$$

Using $r = ck/h$, formula (12) can be simplified to obtain a difference formula for the improved numerical approximations in the second row:

$$u_{i,2} = (1 - r^2) f_i + kg_i + \frac{r^2}{2}(f_{i+1} + f_{i-1}) \tag{13}$$

for $i = 2, 3, \ldots, n-1$.

D'Alembert's Solution

The French mathematician Jean Le Rond d'Alembert (1717–1783) discovered that

$$u(x, t) = F(x + ct) + G(x - ct) \tag{14}$$

is a solution to the wave equation (1) over the interval $0 \le x \le a$, provided that F', F'', G', and G'' all exist and F and G have period $2a$ and obey the relationships $F(-z) = -F(z)$, $F(z + 2a) = F(z)$, $G(-z) = -G(z)$, and $G(z + 2a) = G(z)$ for all z. We can check this out by direct substitution. The second-order partial derivatives of the solution (14) are

$$u_{tt}(x,t) = c^2F''(x+ct) + c^2G''(x-ct), \tag{15}$$

$$u_{xx}(x,t) = F''(x+ct) + G''(x-ct). \tag{16}$$

Substitution of these quantities into (1) produces the desired relationship:

$$\begin{aligned} u_{tt}(x,t) &= c^2F''(x+ct) + c^2G''(x-ct) \\ &= c^2(F''(x+ct) + G''(x-ct)) \\ &= c^2u_{xx}(x,t). \end{aligned}$$

The particular solution that has the boundary values $u(x,0) = f(x)$ and $u_t(x,0) = 0$ requires that $F(x) = G(x) = f(x)/2$ and is left for the reader to verify.

Two Exact Rows Given

The accuracy of the numerical approximations produced by the equations in (7) depends on the truncation errors in the formulas used to convert the partial differential equation into a difference equation. Although it is unlikely to know values of the exact solution for the second row of the grid, if such knowledge were available, using the increment $k = ch$ along the t-axis will generate an exact solution at all the other points throughout the grid.

Theorem 10.1. Assume that the two rows of values $u_{i,1} = u(x_i, 0)$ and $u_{i,2} = u(x_i, k)$, for $i = 1, 2, \ldots, n$, are the exact solutions to the wave equation (1). If the step size $k = h/c$ is chosen along the t-axis, then $r = 1$ and formula (7) becomes

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}. \tag{17}$$

Furthermore, the finite-difference solutions produced by (17) throughout the grid are exact solution values to the differential equation (neglecting computer round-off error).

Proof. Use d'Alembert's solution and the relation $ck = h$. The calculation $x_i - ct_j = (i-1)h - c(j-1)k = (i-1)h - (j-1)h = (i-j)h$ and a similar one producing $x_i + ct_j = (i+j-2)h$ are used in equation (14) to produce the following special form of $u_{i,j}$:

$$u_{i,j} = F((i-j)h) + G((i+j-2)h) \tag{18}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Applying this formula to the terms

$u_{i+1,j}$, $u_{i-1,j}$, and $u_{i,j-1}$ on the right side of (17) yields

$$\begin{aligned}
u_{i+1,j} &+ u_{i-1,j} - u_{i,j-1} \\
&= F((i+1-j)h) + F((i-1-j)h) \\
&\quad - F((i-(j-1))h) + G((i+1+j-2)h) \\
&\quad + G((i-1+j-2)h) - G((i+j-1-2)h) \\
&= F((i-(j+1))h) + G((i+j+1-2)h) = u_{i,j+1},
\end{aligned}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. •

Warning. Theorem 10.1 does not guarantee that the numerical solutions are exact when numerical calculations based on (9) and (13) are used to construct approximations $u_{i,2}$ in the second row. Indeed, truncation error will be introduced if $u_{i,2} \neq u(x_i, k)$ for some i, where $1 \leq i \leq n$. This is why we endeavor to obtain the best possible values for the second row by using the second-order Taylor approximations in equation (13).

Example 10.1. Use the finite-difference method to solve the wave equation for a vibrating string:

$$u_{tt}(x,t) = 4u_{xx}(x,t) \quad \text{for } 0 < x < 1 \text{ and } 0 < t < 0.5, \tag{19}$$

with the boundary conditions

$$\begin{aligned}
&u(0,t) = 0 \quad \text{and} \quad u(1,t) = 0 && \text{for } 0 \leq t \leq 0.5, \\
&u(x,0) = f(x) = \sin(\pi x) + \sin(2\pi x) && \text{for } 0 \leq x \leq 1, \\
&u_t(x,0) = g(x) = 0 && \text{for } 0 \leq x \leq 1.
\end{aligned} \tag{20}$$

For convenience we choose $h = 0.1$ and $k = 0.05$. Since $c = 2$, this yields $r = ck/h = 2(0.05)/0.1 = 1$. Since $g(x) = 0$ and $r = 1$, formula (13) for creating the second row is

$$u_{i,2} = \frac{f_{i-1} + f_{i+1}}{2} \quad \text{for } i = 2, 3, \ldots, 9. \tag{21}$$

Substituting $r = 1$ into equation (7) gives the simplified difference equation

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}. \tag{22}$$

Applying formulas (21) and (22) successively to generate rows will produce the approximations to $u(x,t)$ given in Table 10.1 for $0 < x_i < 1$ and $0 \leq t_j \leq 0.50$.

The numerical values in Table 10.1 agree to more than six decimal places of accuracy with those obtained with the analytic solution

$$u(x,t) = \sin(\pi x)\cos(2\pi t) + \sin(2\pi x)\cos(4\pi t).$$

Table 10.1 Solution of the Wave Equation (19) with Boundary Conditions (20)

t_j	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
0.00	0.896802	1.538842	1.760074	1.538842	1.000000	0.363271	−0.142040	−0.363271	−0.278768
0.05	0.769421	1.328438	1.538842	1.380037	0.951056	0.428980	0.000000	−0.210404	−0.181636
0.10	0.431636	0.769421	0.948401	0.951056	0.809017	0.587785	0.360616	0.181636	0.068364
0.15	0.000000	0.051599	0.181636	0.377381	0.587785	0.740653	0.769421	0.639384	0.363271
0.20	−0.380037	−0.587785	−0.519421	−0.181636	0.309017	0.769421	1.019421	0.951056	0.571020
0.25	−0.587785	−0.951056	−0.951056	−0.587785	0.000000	0.587785	0.951056	0.951056	0.587785
0.30	−0.571020	−0.951056	−1.019421	−0.769421	−0.309017	0.181636	0.519421	0.587785	0.380037
0.35	−0.363271	−0.639384	−0.769421	−0.740653	−0.587785	−0.377381	−0.181636	−0.051599	0.000000
0.40	−0.068364	−0.181636	−0.360616	−0.587785	−0.809017	−0.951056	−0.948401	−0.769421	−0.431636
0.45	0.181636	0.210404	0.000000	−0.428980	−0.951056	−1.380037	−1.538842	−1.328438	−0.769421
0.50	0.278768	0.363271	0.142040	−0.363271	−1.000000	−1.538842	−1.760074	−1.538842	−0.896802

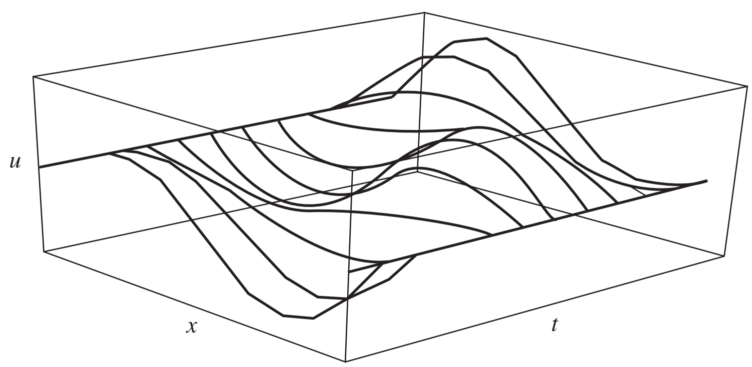

Figure 10.6 The vibrating string for equations (19) and (20).

A three-dimensional presentation of the data in Table 10.1 is given in Figure 10.6. ∎

Example 10.2. Use the finite-difference method to solve the wave equation for a vibrating string:

(23) $$u_{tt}(x,t) = 4u_{xx}(x,t) \quad \text{for } 0 < x < 1 \text{ and } 0 < t < 0.5,$$

with the boundary conditions

(24) $$\begin{aligned} u(0,t) &= 0 \quad \text{and} \quad u(1,t) = 0 \quad \text{for } 0 \le t \le 1, \\ u(x,0) &= f(x) = \begin{cases} x & \text{for } 0 \le x \le \frac{3}{5} \\ 1.5 - 1.5x & \text{for } \frac{3}{5} \le x \le 1, \end{cases} \\ u_t(x,0) &= g(x) = 0 \quad \text{for } 0 < x < 1. \end{aligned}$$

For convenience we choose $h = 0.1$ and $k = 0.05$. Since $c = 2$, this again yields $r = 1$. Applying formulas (21) and (22) successively to generate rows will produce the

Table 10.2 Solution of the Wave Equation (23) with Boundary Conditions (24)

t_j	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
0.00	0.100	0.200	0.300	0.400	0.500	0.600	0.450	0.300	0.150
0.05	0.100	0.200	0.300	0.400	0.500	0.475	0.450	0.300	0.150
0.10	0.100	0.200	0.300	0.400	0.375	0.350	0.325	0.300	0.150
0.15	0.100	0.200	0.300	0.275	0.250	0.225	0.200	0.175	0.150
0.20	0.100	0.200	0.175	0.150	0.125	0.100	0.075	0.050	0.025
0.25	0.100	0.075	0.050	0.025	0.000	−0.025	−0.050	−0.075	−0.100
0.30	−0.025	−0.050	−0.075	−0.100	−0.125	−0.150	−0.175	−0.200	−0.100
0.35	−0.150	−0.175	−0.200	−0.225	−0.250	−0.275	−0.300	−0.200	−0.100
0.40	−0.150	−0.300	−0.325	−0.350	−0.375	−0.400	−0.300	−0.200	−0.100
0.45	−0.150	−0.300	−0.450	−0.475	−0.500	−0.400	−0.300	−0.200	−0.100
0.50	−0.150	−0.300	−0.450	−0.600	−0.500	−0.400	−0.300	−0.200	−0.100

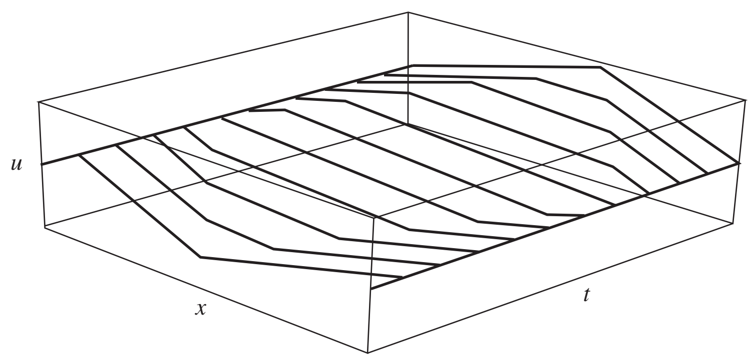

Figure 10.7 The vibrating string for equations (23) and (24).

approximations to $u(x, t)$ given in Table 10.2 for $0 \leq x_i \leq 1$ and $0 \leq t_j \leq 0.50$. A three-dimensional presentation of the data in Table 10.2 is given in Figure 10.7. ■

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